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Lie-central derivations, Lie-centroids and Lie-stem Leibniz algebras

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Abstract. In this paper, we introduce the notion of a Lie-derivation. This concept generalizes derivations for non-Lie Leibniz algebras. We study these Lie-derivations in the case where their image is contained in the Lie-center, and call them Lie-central derivations. We provide a characterization of Lie-stem Leibniz algebras by their Lie-central derivations, and prove several properties of the Lie algebra of Lie-central derivations for Lie-nilpotent Leibniz algebras of class 2. We also introduce ID_*-Lie-derivations. An ID_*-Lie-derivation of a Leibniz algebra $\mathfrak{g}$ is a Lie-derivation of $\mathfrak{g}$ in which the image is contained in the second term of the lower Lie-central series of $\mathfrak{g}$, and which vanishes on Lie-central elements. We provide an upper bound for the dimension of the Lie algebra $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ of ID_*-Lie-derivation of $\mathfrak{g}$, and prove that the sets $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ and $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$ are isomorphic for any two Lie-isoclinic Leibniz algebras $\mathfrak{g}$ and $\mathfrak{q}$.

1. Introduction

Studies such as the work of Dixmier [13], Leger [16] and Tôgô [20]–[23] about the structure of a Lie algebra $\mathcal{L}$ and its relationship with the properties of the Lie algebra of derivations of $\mathcal{L}$ have been conducted by several authors. A classical problem concerning the algebra of derivations is to determine necessary and sufficient conditions under which subalgebras of the algebra of derivations coincide. For example, the coincidence of the subalgebra of central derivations with

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the algebra of derivations of a Lie algebra was studied in [21]. Also, centroids play important roles in the study of extended affine Lie algebras [2], in the investigations of the Brauer groups and division algebras, in the classification of algebras or in the structure theory of algebras. Almost inner derivations arise in many contexts of algebra, number theory or geometry, for instance, they play an important role in the study of isospectral deformations of compact solvmanifolds [15]; the paper [6] is dedicated to studying almost inner derivations of Lie algebras.

Our aim in this paper is to conduct an analogue study by investigating various concepts of derivations on Leibniz algebras. Our study relies on the relative notions of these derivations; derivations relative to the *Liezation functor* $(-)_{\mathsf{Lie}}$: $\mathsf{Leib} \to \mathsf{Lie}$, which assigns to a Leibniz algebra $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{_{\rm Lie}}$, where Leib denotes the category of Leibniz algebras, and Lie denotes the category of Lie algebras.

The approached properties are closely related to the relative notions of central extension in a semi-abelian category with respect to a Birkhoff subcategory (see [11] and [14]). A recent research line deals with the development of absolute properties of Leibniz algebras (absolute are the usual properties and it means relative to the abelianization functor) in the relative setting (with respect to the Liezation functor); in general, absolute properties have the corresponding relative ones, but not all absolute properties immediately hold in the relative case, so new requirements are needed as it can be seen in the papers [3]–[5], [8], [10] and [19].

In order to develop a systematic study of derivations in the relative setting, we organize the paper as follows. In Section 2, we provide some background on relative notions with respect to the Liezation functor. We define the sets of Lie-derivations $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ and central Lie-derivations $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ for a non-Lie Leibniz algebra $\mathfrak{g}$. It is worth mentioning that the absolute derivations are also Lie-derivations. In Section 3, we characterize Lie-stem Leibniz algebras using their Lie-central derivations. Using Lie-isoclinism, we prove several results on the Lie algebra of Lie-central derivations of Lie-nilpotent Leibniz algebras of class two. Specifically, we prove that $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is abelian if and only if $Z_{\mathsf{Lie}}(\mathfrak{g}) = \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$, under the assumption that $\mathfrak{g}$ is a finite dimensional Lie-nilpotent Leibniz algebra of class 2. In Section 4, we define the Lie-centroid $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$ of $\mathfrak{g}$ and prove several of its basic properties. In particular, we study its relationship with the Lie-algebra $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. In Section 5, we study the set $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ of ID_*-Lie-derivations of a Leibniz algebra $\mathfrak{g}$ and its subalgebra $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ of almost inner Lie-derivations of $\mathfrak{g}$. Similarly to the result of Tôgô [22] on derivations of Lie algebras, we provide necessary and sufficient conditions on a finite dimensional Leibniz algebra $\mathfrak{g}$ for the subalgebras $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ and $\mathsf{ID}_*(\mathfrak{g})$ to be equal. We also prove that if two Leibniz algebras are Lie-isoclinic, then their sets of ID_*-Lie-derivations are isomorphic. This isomorphism also holds for their sets of almost inner Lie-derivations. We establish several results on almost inner Lie-derivations, similarly to the Lie algebra case [6]. Finally, we provide an upper bound of the dimension of $\mathsf{ID}_*(\mathfrak{g})$ by means of the dimension of $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$.

2. Preliminaries on Leibniz algebras

Let $\mathbb{K}$ be a fixed field of characteristic different from 2. Throughout the paper, all vector spaces and tensor products are considered over $\mathbb{K}$.

A *Leibniz algebra* [17]–[18] is a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, usually called the *Leibniz bracket* of $\mathfrak{g}$, satisfying the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x, y, z \in \mathfrak{g}.$$

In fact, this definition corresponds to the notion of right Leibniz algebra, which means that the right operator $R_x \colon \mathfrak{g} \to \mathfrak{g}$, $R_x(y) = [y,x]$, is a derivation. Correspondingly, the notion of left Leibniz algebra is associated with the fact that the left operation $L_x \colon \mathfrak{g} \to \mathfrak{g}$, $L_x(y) = [x,y]$, is a derivation, which means that the identity $[[x,y],z] = [x,[y,z]] - [y,[x,z]]$ holds for all $x, y, z \in \mathfrak{g}$. The passage from the right to the left Leibniz algebra can be done considering the new bracket operation $[x,y]' = [y,x]$. Throughout this paper, we only consider right Leibniz algebras.

A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$ is said to be a *left (resp. right) ideal* of $\mathfrak{g}$ if $[h,g] \in \mathfrak{h}$ (resp. $[g,h] \in \mathfrak{h}$), for all $h \in \mathfrak{h}$, $g \in \mathfrak{g}$. If $\mathfrak{h}$ is both a left and a right ideal, then $\mathfrak{h}$ is called a *two-sided ideal* of $\mathfrak{g}$. In this case, $\mathfrak{g}/\mathfrak{h}$ naturally inherits a Leibniz algebra structure.

Given a Leibniz algebra $\mathfrak{g}$, we denote by $\mathfrak{g}^{\mathrm{ann}}$ the subspace of $\mathfrak{g}$ spanned by all elements of the form $[x,x]$, $x \in \mathfrak{g}$. It is clear that the quotient $\mathfrak{g}_{\mathrm{Lie}} = \mathfrak{g}/\mathfrak{g}^{\mathrm{ann}}$ is a Lie algebra. This defines the so-called *Liezation functor* $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which assigns to a Leibniz algebra $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$. Moreover, the canonical epimorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\mathrm{Lie}}$ is universal among all homomorphisms from $\mathfrak{g}$ to a Lie algebra, implying that the Liezation functor is left adjoint to the inclusion functor $\mathsf{Lie} \hookrightarrow \mathsf{Leib}$.

Given a Leibniz algebra $\mathfrak{g}$, we define the bracket

$$[-,-]_{\mathrm{lie}} : \mathfrak{g} \to \mathfrak{g}, \quad \text{by } [x,y]_{\mathrm{lie}} = [x,y] + [y,x], \quad \text{for } x, y \in \mathfrak{g}.$$

Let $\mathfrak{m}$, $\mathfrak{n}$ be two-sided ideals of a Leibniz algebra $\mathfrak{g}$. The following notions come from [10], which were derived from [11].

The Lie-*commutator* of $\mathfrak{m}$ and $\mathfrak{n}$ is the two-sided ideal of $\mathfrak{g}$:

$$[\mathfrak{m},\mathfrak{n}]_{\mathsf{Lie}} = \langle \{[m,n]_{\mathrm{lie}}, m \in \mathfrak{m}, n \in \mathfrak{n}\}\rangle.$$

The Lie-*center* of the Leibniz algebra $\mathfrak{g}$ is the two-sided ideal

$$Z_{\mathsf{Lie}}(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [g,z]_{\mathrm{lie}} = 0 \text{ for all } g \in \mathfrak{g}\}.$$

The Lie-*centralizer* of $\mathfrak{m}$ and $\mathfrak{n}$ over $\mathfrak{g}$ is

$$C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m},\mathfrak{n}) = \{g \in \mathfrak{g} \mid \ [g,m]_{\mathrm{lie}} \in \mathfrak{n}, \text{ for all } m \in \mathfrak{m}\}\ .$$

Obviously, $C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{g},0) = Z_{\mathsf{Lie}}(\mathfrak{g})$.

The right-center of a Leibniz algebra $\mathfrak{g}$ is the two-sided ideal $Z^r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [y,x] = 0 \text{ for all } y \in \mathfrak{g}\}$. The left-center of a Leibniz algebra $\mathfrak{g}$ is the set $Z^l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \text{ for all } y \in \mathfrak{g}\}$, which might not even be a subalgebra. $Z(\mathfrak{g}) = Z^l(\mathfrak{g}) \cap Z^r(\mathfrak{g})$ is called the center of $\mathfrak{g}$, which is a two-sided ideal of $\mathfrak{g}$.

Definition 2.1 ([10]). Let $\mathfrak{n}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{g}$. The lower Lie-central series of $\mathfrak{g}$ relative to $\mathfrak{n}$ is the sequence

$$\cdots \trianglelefteq \gamma_i^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) \trianglelefteq \cdots \trianglelefteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) \trianglelefteq \gamma_1^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n})$$

of two-sided ideals of $\mathfrak{g}$ defined inductively by

$$\gamma_1^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) = \mathfrak{n} \quad \text{and} \quad \gamma_i^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) = [\gamma_{i-1}^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}),\mathfrak{g}]_{\mathsf{Lie}}, \quad i \geq 2.$$

We use the notation $\gamma_i^{\mathsf{Lie}}(\mathfrak{g})$ instead of $\gamma_i^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{g})$, $1 \leq i \leq n$.

If $\varphi : \mathfrak{g} \to \mathfrak{q}$ is a homomorphism of Leibniz such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ is a two-sided ideal of $\mathfrak{g}$, and $\mathfrak{n}$ a two-sided ideal of $\mathfrak{q}$, then $\varphi(\gamma_i^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{m})) \subseteq \gamma_i^{\mathsf{Lie}}(\mathfrak{q},\mathfrak{n})$, $i \geq 1$.

Definition 2.2. The Leibniz algebra $\mathfrak{g}$ is said to be Lie-nilpotent relative to $\mathfrak{n}$ of class c if $\gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) = 0$ and $\gamma_c^{\mathsf{Lie}}(\mathfrak{g},\mathfrak{n}) \neq 0$.

Definition 2.3 ([10]). The upper Lie-central series of a Leibniz algebra $\mathfrak{g}$ is the sequence of two-sided ideals, called i-Lie centers, $i = 0,1,2,\dots,$

$$\mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{g}) \trianglelefteq \mathcal{Z}_1^{\mathsf{Lie}}(\mathfrak{g}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g}) \trianglelefteq \cdots$$

defined inductively by

$$\mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{g}) = 0 \quad \text{and} \quad \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g}) = C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{g}, \mathcal{Z}_{i-1}^{\mathsf{Lie}}(\mathfrak{g})), \quad i \geq 1.$$

Theorem 2.4 ([10, Theorem 4])**.** *A Leibniz algebra $\mathfrak{g}$ is* Lie*-nilpotent of class c if and only if $Z_c^{\mathsf{Lie}}(\mathfrak{g}) = \mathfrak{g}$ and $Z_{c-1}^{\mathsf{Lie}}(\mathfrak{g}) \neq \mathfrak{g}$.*

Definition 2.5 ([8, Definition 2.8]). Let $\mathfrak{m}$ be a subset of a Leibniz algebra $\mathfrak{g}$. The Lie-normalizer of $\mathfrak{m}$ is the subset of $\mathfrak{g}$:

$$N_{\mathfrak{g}}(\mathfrak{m}) = \{g \in \mathfrak{g} \mid [g, m], [m, g] \in \mathfrak{m}, \text{for all } m \in \mathfrak{m}\}.$$

Definition 2.6 ([10, Proposition 1]). An exact sequence of Leibniz algebras $0 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ is said to be a Lie-*central* extension if $[\mathfrak{g}, \mathfrak{n}]_{\mathsf{Lie}} = 0$, equivalently, $\mathfrak{n} \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$.

Definition 2.7. A linear map $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $(\mathfrak{g}, [-,-])$ is said to be a Lie-derivation if for all $x, y \in \mathfrak{g}$, the following condition holds:

$$d([x, y]_{\mathrm{lie}}) = [d(x), y]_{\mathrm{lie}} + [x, d(y)]_{\mathrm{lie}}.$$

We denote by $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ the set of all Lie-derivations of a Leibniz algebra $\mathfrak{g}$, which can be equipped with a structure of Lie algebra by means of the usual bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, for all $d_1, d_2 \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$.

Example 2.8. The absolute derivations, that is the linear maps $d\colon \mathfrak{g} \to \mathfrak{g}$ such that $d([x, y]) = [d(x), y] + [x, d(y)]$, are also Lie-derivations, since:

$$d([x, y]_{\mathrm{lie}}) = d([x, y] + [y, x]) = [d(x), y]_{\mathrm{lie}} + [x, d(y)]_{\mathrm{lie}}, \quad \text{for all } x, y \in \mathfrak{g}. \tag{1}$$

In particular, for a fixed $x \in \mathfrak{g}$, the inner derivation $R_x\colon \mathfrak{g} \to \mathfrak{g}, R_x(y) = [y, x]$, for all $y \in \mathfrak{g}$, is a Lie-derivation, so it gives rise to the following identity:

$$[[y, z]_{\mathrm{lie}}, x] = [[y, x], z]_{\mathrm{lie}} + [y, [z, x]]_{\mathrm{lie}}, \quad \text{for all } x, y \in \mathfrak{g}.$$

However, there are Lie-derivations which are not derivations. For instance, every linear map $d\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie-derivation for any Lie algebra $\mathfrak{g}$, but it is not a derivation in general.

Definition 2.9. A Lie-derivation $d : \mathfrak{g} \to \mathfrak{g}$ of a Leibniz algebra $\mathfrak{g}$ is said to be a Lie-central derivation if its image is contained in the Lie-center of $\mathfrak{g}$.

Remark 2.10. The absolute notion corresponding to Definition 2.9 is the so-called central derivation, that is a derivation $d : \mathfrak{g} \to \mathfrak{g}$ whose image is contained in the center of $\mathfrak{g}$. Obviously, every central derivation is a Lie-central derivation. However, the converse is not true as the following example shows. Let $\mathfrak{g}$ be the two-dimensional Leibniz algebra with basis $\{e, f\}$ and bracket operation given by $[e, f] = -[f, e] = e$ [12]. Then the inner derivation R_e is a Lie-central derivation, but it is not central in general.

We denote the set of all Lie-central derivations of a Leibniz algebra $\mathfrak{g}$ by $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Obviously, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is a subalgebra of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, and every element of $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ annihilates $\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$. One has that $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) = C_{\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})}((\mathsf{R}+\mathsf{L})(\mathfrak{g}))$, where $\mathsf{L}(\mathfrak{g}) = \{L_x \mid x \in \mathfrak{g}\}$, L_x denotes the left multiplication operator $L_x(y) = [x,y]$, $\mathsf{R}(\mathfrak{g}) = \{R_x \mid x \in \mathfrak{g}\}$ and $C_{\mathfrak{g}}(\mathfrak{m}) = \{x \in \mathfrak{g} \mid [x,y] = 0 = [y,x], \text{for all } y \in \mathfrak{m}\}$, the absolute centralizer of an ideal $\mathfrak{m}$ over the Leibniz algebra $\mathfrak{g}$.

Let A and B be two Leibniz algebras, and denote by $T(A,B)$ the set of all linear transformations from A to B. Clearly, $T(A,B)$ endowed with the bracket $[f,g](x) = [f(x), g(x)]$ is an abelian Leibniz algebra if B is also an abelian Leibniz algebra.

Consider the Lie-central extensions $(g) : 0 \to \mathfrak{n} \overset{\chi}{\to} \mathfrak{g} \overset{\pi}{\to} \mathfrak{q} \to 0$ and $(g_i) : 0 \to \mathfrak{n}_i \overset{\chi_i}{\to} \mathfrak{g}_i \overset{\pi_i}{\to} \mathfrak{q}_i \to 0, i = 1,2$.

Let $C : \mathfrak{q} \times \mathfrak{q} \to [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ be given by $C(q_1,q_2) = [g_1,g_2]_{\mathrm{lie}}$, where $\pi(g_j) = q_j, j = 1,2$, the Lie-commutator map associated to the extension (g). In a similar way are defined the Lie-commutator maps C_i corresponding to the extensions $(g_i), i = 1,2$.

Note that if $\mathfrak{q}$ is a Lie algebra, then $\pi([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = 0$, hence $[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{n} \equiv \chi(\mathfrak{n})$.

Definition 2.11 ([3, Definition 3.1]). The Lie-central extensions (g_1) and (g_2) are said to be Lie-isoclinic when there exist isomorphisms $\eta : \mathfrak{q}_1 \to \mathfrak{q}_2$ and $\xi : [\mathfrak{g}_1,\mathfrak{g}_1]_{\mathsf{Lie}} \to [\mathfrak{g}_2,\mathfrak{g}_2]_{\mathsf{Lie}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{q}_1 \times \mathfrak{q}_1 \arrow[r, "C_1"] \arrow[d, "\eta\times\eta"'] & {[\mathfrak{g}_1,\mathfrak{g}_1]_{\mathsf{Lie}}} \arrow[d, "\xi"] \\
\mathfrak{q}_2 \times \mathfrak{q}_2 \arrow[r, "C_2"] & {[\mathfrak{g}_2,\mathfrak{g}_2]_{\mathsf{Lie}}}
\end{tikzcd} \tag{2}$$

The pair (η,ξ) is called a Lie-isoclinism from (g_1) to (g_2), and it will be denoted by $(\eta,\xi) : (g_1) \to (g_2)$.

Let $\mathfrak{g}$ be a Leibniz algebra. Then we can construct the following Lie-central extension:

$$(e_g) : 0 \to Z_{\mathsf{Lie}}(\mathfrak{g}) \to \mathfrak{g} \overset{pr_{\mathfrak{g}}}{\to} \mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g}) \to 0. \tag{3}$$

Definition 2.12 ([3, Definition 3.3]). Let $\mathfrak{g}$ and $\mathfrak{q}$ be Leibniz algebras. Then $\mathfrak{g}$ and $\mathfrak{q}$ are said to be Lie-isoclinic when (e_g) and (e_q) are Lie-isoclinic Lie-central extensions.

A Lie-isoclinism (η,ξ) from (e_g) to (e_q) is also called a Lie-isoclinism from $\mathfrak{g}$ to $\mathfrak{q}$, denoted by $(\eta,\xi) : \mathfrak{g} \sim \mathfrak{q}$.

Proposition 2.13 ([3, Proposition 3.4]). *For a* Lie-*isoclinism* $(\eta, \xi) : (g_1) \sim (g_2)$*, the following statements hold:*

(a) η *induces an isomorphism* $\eta' : \mathfrak{g}_1/Z_{\mathsf{Lie}}(\mathfrak{g}_1) \to \mathfrak{g}_2/Z_{\mathsf{Lie}}(\mathfrak{g}_2)$*, and* (η', ξ) *is a* Lie-*isoclinism from* $\mathfrak{g}_1$ *to* $\mathfrak{g}_2$.

(b) $\chi_1(\mathfrak{n}_1) = Z_{\mathsf{Lie}}(\mathfrak{g}_1)$ *if and only if* $\chi_2(\mathfrak{n}_2) = Z_{\mathsf{Lie}}(\mathfrak{g}_2)$.

Definition 2.14 ([19, Definition 4]). A Lie-stem Leibniz algebra is a Leibniz algebra $\mathfrak{g}$ such that $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$.

Theorems 1 and 2 in [19] prove that every Lie-isoclinic family of Leibniz algebras contains at least one Lie-stem Leibniz algebra, which is of minimal dimension if it has finite dimension.

3. Lie-stem Leibniz algebras and Lie-central derivations

Proposition 3.1. *If* $\mathfrak{g}$ *is a* Lie-*stem Leibniz algebra, then* $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ *is an abelian Lie algebra.*

Proof. Since $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is a subalgebra of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, it is enough to show that $[d_1, d_2] = 0$ for all $d_1, d_2 \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. First, we notice that if $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, then $d([x, y]_{\mathrm{lie}}) = 0$ for all $x, y \in \mathfrak{g}$, since $d(x), d(y) \in Z_{\mathsf{Lie}}(\mathfrak{g})$. So in particular, $d(Z_{\mathsf{Lie}}(\mathfrak{g})) = 0$, since $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, as $\mathfrak{g}$ is a Lie-stem Leibniz algebra. Now let $d_1, d_2 \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ and $x \in \mathfrak{g}$. Then $d_1(x), d_2(x) \in Z_{\mathsf{Lie}}(\mathfrak{g})$, which implies that $[d_1, d_2](x) = d_1(d_2(x)) - d_2(d_1(x)) = 0$. Hence $[d_1, d_2] = 0$. □

The converse of the above result is not true in general. Indeed, let $\mathfrak{g}$ be any Lie algebra. Then $Z_{\mathsf{Lie}}(\mathfrak{g}) = \mathfrak{g}$, and so $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is an abelian Lie algebra. However, $\mathfrak{g}$ is not a Lie-stem Leibniz algebra, since $Z_{\mathsf{Lie}}(\mathfrak{g}) = \mathfrak{g} \not\subseteq 0 = [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$.

Proposition 3.2. *Let* $\mathfrak{g}$ *be a* Lie-*nilpotent finite dimensional Leibniz algebra such that* $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \neq 0$*. Then* $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ *is abelian if and only if* $\mathfrak{g}$ *is a* Lie-*stem Leibniz algebra.*

Proof. We only need to prove the converse of Proposition 3.1. Assume that $\mathfrak{g}$ is not a Lie-stem Leibniz algebra. Then, there is some $z_1 \in Z_{\mathsf{Lie}}(\mathfrak{g})$ such that $z_1 \notin [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Since $\mathfrak{g}$ is a Lie-nilpotent Leibniz algebra and $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \neq 0$, it follows that $Z_{\mathsf{Lie}}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} \neq 0$. Now, let $H := \langle z_1 \rangle^{\perp}$ be the complement of the subspace spanned by z_1, and let $z_2 \in Z_{\mathsf{Lie}}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, $z_2 \neq 0$. Consider d_1 (d_2, respectively) as the linear transformation of $\mathfrak{g}$ vanishing on H and mapping z_1 to z_1 (z_1 to z_2, respectively). Clearly, d_1 and d_2 are Lie-central derivations, and

d_1 and d_2 do not commute, since $[d_1, d_2](z_1) = d_1(d_2(z_1)) - d_2(d_1(z_1)) = -z_2 \neq 0$. Therefore, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is not abelian. This completes the proof. $\square$

Lemma 3.3. *Let (η, ξ) be a* Lie*-isoclinism between the Leibniz algebras $\mathfrak{g}$ and $\mathfrak{q}$. If $\mathfrak{g}$ is a* Lie*-stem Leibniz algebra, then ξ maps $Z_{\mathsf{Lie}}(\mathfrak{g})$ onto $Z_{\mathsf{Lie}}(\mathfrak{q}) \cap [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$.*

Proof. Since $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$, an element z of $Z_{\mathsf{Lie}}(\mathfrak{g})$ can be written as $z = \sum\limits_{i=1}^{n} \lambda_i\, [x_i, y_i]_{\mathrm{lie}}$, with $\lambda_i \in \mathbb{K}$ and $x_i, y_i \in \mathfrak{g}, i = 1, \ldots, n$.

Let $\eta' : \mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow \mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q})$, $\eta'\,(x_i + Z_{\mathsf{Lie}}(\mathfrak{g})) = \eta(x_i) + Z_{\mathsf{Lie}}(\mathfrak{q})$ and $\eta'\,(y_i + Z_{\mathsf{Lie}}(\mathfrak{g})) = \eta(y_i) + Z_{\mathsf{Lie}}(\mathfrak{q}), i = 1, \ldots, n$, be the isomorphism provided by [3, Proposition 3.4]. Then

$$\begin{aligned}\xi\,(z) + Z_{\mathsf{Lie}}(\mathfrak{q}) &= \xi\left(\sum_{i=1}^{n} \lambda_i\, [x_i, y_i]_{\mathrm{lie}}\right) + Z_{\mathsf{Lie}}(\mathfrak{q}) = \sum_{i=1}^{n} \lambda_i \xi\, [x_i, y_i]_{\mathrm{lie}} + Z_{\mathsf{Lie}}(\mathfrak{q}) \\ &= \sum_{i=1}^{n} \lambda_i\, [\eta(x_i), \eta(y_i)]_{\mathrm{lie}} + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta'\left(\sum_{i=1}^{n} \lambda_i\, [x_i, y_i]_{\mathrm{lie}} + Z_{\mathsf{Lie}}(\mathfrak{g})\right) = Z_{\mathsf{Lie}}(\mathfrak{q}).\end{aligned}$$

The surjective property can be easily established. $\square$

Proposition 3.4. *Let $\mathfrak{g}$ and $\mathfrak{q}$ be two* Lie*-isoclinic Leibniz algebras, and $\mathfrak{g}$ be a* Lie*-stem Leibniz algebra. Then every $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ induces a* Lie*-central derivation d^* of $\mathfrak{q}$. Moreover, the map $d \mapsto d^*$ is a monomorphism from $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ into $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$.*

Proof. Let (η, ξ) be a Lie-isoclinism between $\mathfrak{g}$ and $\mathfrak{q}$, and let $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Then for any $y \in \mathfrak{q}$, we have $y + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x + Z_{\mathsf{Lie}}(\mathfrak{g}))$ for some $x \in \mathfrak{g}$, since η is bijective. Now consider the map $d^* : \mathfrak{q} \to \mathfrak{q}$ defined by $d^*(y) = \xi(d(x))$, which is well-defined, since $d(Z_{\mathsf{Lie}}(\mathfrak{g})) = 0$ as $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$. Moreover, $d^* \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$, since $d(x) \in Z_{\mathsf{Lie}}(\mathfrak{g})$ and $\xi(d(x)) \in Z_{\mathsf{Lie}}(\mathfrak{q}) \cap [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$ by Lemma 3.3. Observe that d^* is a Lie-derivation, since $d^*([y_1, y_2]_{\mathrm{lie}}) = \xi(d([x_1, x_2]_{\mathrm{lie}})) = \xi([d(x_1), x_2]_{\mathrm{lie}} + [x_1, d(x_2)]_{\mathrm{lie}}) = \xi(0 + 0) = 0$ and $[y_1, d^*(y_2)]_{\mathrm{lie}} + [d^*(y_1), y_2]_{\mathrm{lie}} = 0$, since $d^*(y_1)$, $d^*(y_2) \in Z_{\mathsf{Lie}}(\mathfrak{q})$.

Clearly, the map $\phi : d \to d^*$ is linear and one-to-one, since ξ an isomorphism. To show that ϕ is compatible with the Lie-bracket, let $d_1, d_2 \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Then for $i, j = 1, 2$, we have $d_i(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$ and $d_j([\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}) = 0$. As a consequence, on the one hand $[d_1, d_2] = d_1 d_2 - d_2 d_1 = 0$, and thus $[d_1, d_2]^* = 0$, as ξ is an isomorphism. On the other hand, $d_i^*(\mathfrak{q}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{q}) \cap [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$. So $d_j^*(d_i^*(\mathfrak{q})) = 0$, by definition of d_j^*, and thus $[d_1^*, d_2^*] = d_1^* d_2^* - d_2^* d_1^* = 0$. Therefore, $\phi([d_1, d_2]) = [\phi(d_1), \phi(d_2)]$. This completes the proof. $\square$

Lemma 3.5. *For any* Lie*-stem Leibniz algebra* $\mathfrak{g}$*, there is a Lie algebra isomorphism* $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) \cong T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$.

Proof. Let $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, then $d(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$, and thus $d([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = 0$. So d induces the map $\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}} \xrightarrow{\alpha_d} Z_{\mathsf{Lie}}(\mathfrak{g})$ defined by $\alpha_d(x+[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = d(x)$. Now define the map $\beta\colon \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$ by $\beta(d) = \alpha_d$. Clearly, β is a linear map, which is one-to-one by definition of α_d.

β is onto, since for a given $d^* \in T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$, there exists a linear map $d : \mathfrak{g} \to Z_{\mathsf{Lie}}(g)$, $d = d^* \circ \pi$, where $\pi : \mathfrak{g} \to \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}$ is the canonical projection, such that $\beta(d) = d^*$. Finally, $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, since $d([x,y]_{\text{lie}}) = d^*([\pi(x),\pi(y)]_{\text{lie}}) = d^*(\overline{0}) = 0$. On the other hand, $[d(x),y]_{\text{lie}} + [x,d(y)]_{\text{lie}} = [d^*(\pi(x)),y]_{\text{lie}} + [x,d^*(\pi(y))]_{\text{lie}} = 0$, since $d^*(\pi(x)), d^*(\pi(y)) \in Z_{\mathsf{Lie}}(\mathfrak{g})$. To finish, we show that $\beta([d_1,d_2]) = [\beta(d_1),\beta(d_2)]$ for all $d_1,d_2 \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Indeed, let $x \in \mathfrak{g}$. It is clear that $\beta([d_1,d_2])(\pi(x)) = \alpha_{[d_1,d_2]}(\pi(x)) = [d_1,d_2](x) = 0$, since $d_1(\mathfrak{g}), d_2(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}$ and $d_1([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = d_2([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = 0$. On the other hand, $[\beta(d_1),\beta(d_2)](\pi(x)) = [\alpha_{d_1},\alpha_{d_2}](\pi(x)) = \alpha_{d_1}(d_2(x)) - \alpha_{d_2}(d_1(x)) = 0$, since $\alpha_{d_1}([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}) = 0 = \alpha_{d_2}([\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}})$. Hence $\beta([d_1,d_2]) = [\beta(d_1),\beta(d_2)]$. This completes the proof. $\square$

Corollary 3.6. *For any arbitrary Leibniz algebra* $\mathfrak{q}$*, the Lie algebra* $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ *has a central subalgebra* $\mathfrak{n}$ *isomorphic to* $T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$ *for some* Lie*-stem Leibniz algebra* $\mathfrak{g}$ Lie*-isoclinic to* $\mathfrak{q}$*. Moreover, each element of* $\mathfrak{n}$ *sends* $Z_{\mathsf{Lie}}(\mathfrak{q})$ *to the zero subalgebra.*

Proof. By [5, Corollary 4.1], there is a Lie-stem Leibniz algebra $\mathfrak{g}$ that is Lie-isoclinic to $\mathfrak{q}$. Denote this Lie-isoclinism by (η,ξ). Now, by the proof of Proposition 3.4, $\mathfrak{n} := \{d^* \mid d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})\}$ is a subalgebra of $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ isomorphic to $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Moreover, $\mathfrak{n}$ is a central subalgebra of $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$. Indeed, let $d_0 \in \mathfrak{n}$ and $d_1 \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$. Then for any $y \in \mathfrak{q}$, we have by definition, $d_0^*(y) = \xi(d_0(x))$ with $\pi_2(y) = \eta(\pi_1(x))$. So $d_1(d_0^*(y)) = 0$, since $d_0^*(\mathfrak{q}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{q}) \cap [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}$ by Lemma 3.3, and $d_1([\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}) = 0$. Also, $d_0^*(Z_{\mathsf{Lie}}(\mathfrak{q})) = 0$, since η is one-to-one and ξ is a homomorphism. In particular, $d_0^*(d_1(y)) = 0$, since $d_1(\mathfrak{q}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{q})$. Therefore $[d_0^*, d_1] = 0$. Moreover, for any $d_0^* \in \mathfrak{n}$, we have $d_0^*(Z_{\mathsf{Lie}}(\mathfrak{q})) = 0$ as mentioned above. To complete the proof, notice that $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) \cong T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$ due to Lemma 3.5. $\square$

Lemma 3.7. *Let* $\mathfrak{g}$ *and* $\mathfrak{q}$ *be two* Lie*-isoclinic Leibniz algebras. If* $\mathfrak{g}$ *is* Lie*-nilpotent of class* c*, then so is* $\mathfrak{q}$.

Proof. Notice that for all $g \in \mathfrak{g}$ and $x_1, x_2, \ldots, x_i \in \mathfrak{g}$, and setting $\bar{t} := t + Z_{\mathsf{Lie}}(\mathfrak{g})$, $t = g, x_1, x_2, \ldots, x_i$, we have

$$[[[\bar{g}, \bar{x}_1]_{\mathrm{lie}}, \bar{x}_2]_{\mathrm{lie}}, \ldots, \bar{x}_i]_{\mathrm{lie}} = [[[g, x_1]_{\mathrm{lie}}, x_2]_{\mathrm{lie}}, \ldots, x_i]_{\mathrm{lie}} + Z_{\mathsf{Lie}}(\mathfrak{g}).$$

So $g \in \mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{g})$ if and only if $g + Z_{\mathsf{Lie}}(\mathfrak{g}) \in \mathcal{Z}^{\mathsf{Lie}}_{i}(\mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g}))$. Thus $\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{g})/Z_{\mathsf{Lie}}(\mathfrak{g}) = \mathcal{Z}^{\mathsf{Lie}}_{i}(\mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g}))$. If (η, ξ) is the Lie-isoclinism between $\mathfrak{g}$ and $\mathfrak{q}$, as η is an isomorphism, we have

$$\eta(\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{g})/Z_{\mathsf{Lie}}(\mathfrak{g})) = \eta(\mathcal{Z}^{\mathsf{Lie}}_{i}(\mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g}))) = \mathcal{Z}^{\mathsf{Lie}}_{i}(\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q})).$$

It follows that

$$\mathfrak{g}/\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{g}) \cong \frac{\mathfrak{g}/Z_{\mathsf{Lie}}(\mathfrak{g})}{\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{g})/Z_{\mathsf{Lie}}(\mathfrak{g})} \cong \frac{\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q})}{\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{q})/Z_{\mathsf{Lie}}(\mathfrak{q})} \cong \mathfrak{q}/\mathcal{Z}^{\mathsf{Lie}}_{i+1}(\mathfrak{q}).$$

Now, assume that $\mathfrak{g}$ is Lie-nilpotent of class c. Then $\mathcal{Z}^{\mathsf{Lie}}_{c}(\mathfrak{g}) = \mathfrak{g}$. So $\mathfrak{q}/\mathcal{Z}^{\mathsf{Lie}}_{c}(\mathfrak{q}) \cong \mathfrak{g}/\mathcal{Z}^{\mathsf{Lie}}_{c}(\mathfrak{g}) = 0$, implying that $\mathcal{Z}^{\mathsf{Lie}}_{c}(\mathfrak{q}) = \mathfrak{q}$. Also, $\mathfrak{g}/\mathcal{Z}^{\mathsf{Lie}}_{c-1}(\mathfrak{g}) \neq 0 \iff \mathfrak{q}/\mathcal{Z}^{\mathsf{Lie}}_{c-1}(\mathfrak{q}) \neq 0$. Hence $\mathfrak{q}$ is also Lie-nilpotent of class c. □

Corollary 3.8. *Let $\mathfrak{q}$ be a Lie-nilpotent Leibniz algebra of class 2. Then $\mathsf{Der}^{\mathsf{Lie}}_{z}(\mathfrak{q})$ has a central subalgebra isomorphic to $T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}\right)$ containing $(R+L)(\mathfrak{q})$.*

Proof. By [5, Corollary 4.1], there is a Lie-stem Leibniz algebra $\mathfrak{g}$ Lie-isoclinic to $\mathfrak{q}$. Denote this Lie-isoclinism by (η, ξ). Since $\mathfrak{q}$ is a Lie-nilpotent Leibniz algebra of class 2, so is $\mathfrak{g}$, due to to Lemma 3.7. Then $Z_{\mathsf{Lie}}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}} \overset{\xi}{\cong} [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}$, and $\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}} \cong \frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})} \overset{\eta}{\cong} \frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}$. So $T\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{g})\right) \cong T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}\right)$. Therefore $\mathsf{Der}^{\mathsf{Lie}}_{z}(\mathfrak{q})$ has a central subalgebra $\mathfrak{n}$ isomorphic to $T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}\right)$, due to to Corollary 3.6. Moreover, the map $\zeta : \frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})} \to T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}\right)$ defined by $x + Z_{\mathsf{Lie}}(\mathfrak{q}) \mapsto \zeta(x + Z_{\mathsf{Lie}}(\mathfrak{q})) : \frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})} \to [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}$ with $\zeta(x + Z_{\mathsf{Lie}}(\mathfrak{q}))(y + Z_{\mathsf{Lie}}(\mathfrak{q})) = [x,y]_{\mathrm{lie}}$, is a well-defined one-to-one linear map, since for all $x, x' \in \mathfrak{q}$,

$$\begin{aligned} x - x' \in Z_{\mathsf{Lie}}(\mathfrak{q}) &\iff [x - x', y]_{\mathrm{lie}} = 0 \quad \text{for all } y \in \mathfrak{q} \\ &\iff [x,y]_{\mathrm{lie}} = [x',y]_{\mathrm{lie}} \quad \text{for all } y \in \mathfrak{q} \\ &\iff \zeta(\overline{x})(y + Z_{\mathsf{Lie}}(\mathfrak{q})) = \zeta(\overline{x'})(y + Z_{\mathsf{Lie}}(\mathfrak{q})) \quad \text{for all } y \in \mathfrak{q} \\ &\iff \zeta(\overline{x}) = \zeta(\overline{x'}). \end{aligned}$$

Here we use the notation $\overline{x} = x + Z_{\mathsf{Lie}}(\mathfrak{q})$.

Finally, $(R+L)(\mathfrak{q}) = \mathsf{Im}(\zeta) \subseteq T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q},\mathfrak{q}]_{\mathsf{Lie}}\right)$, since $\zeta(\overline{x})(\overline{y}) = [x,y]_{\mathrm{lie}} = [x,y] + [y,x] = L_x(y) + R_x(y)$. □

For any Leibniz algebra $\mathfrak{g}$ with $\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ abelian, we put

$$K(\mathfrak{g}) := \bigcap \mathsf{Ker}\left(f : \mathfrak{g} \to \gamma_2^{\mathsf{Lie}}(\mathfrak{g})\right).$$

Lemma 3.9. *Let $\mathfrak{q}$ be a* Lie*-nilpotent Leibniz algebra of class 2. Then $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = K(\mathfrak{q})$.*

Proof. Let $f : \mathfrak{q} \to \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$ be a homomorphism of Leibniz algebras. Then for all $q_1, q_2 \in \mathfrak{q}$, we have $f([q_1, q_2]_{\mathrm{lie}}) = [f(q_1), f(q_2)]_{\mathrm{lie}} \in [\gamma_2^{\mathsf{Lie}}(\mathfrak{q}), \gamma_2^{\mathsf{Lie}}(\mathfrak{q})]_{\mathsf{Lie}} \subseteq \gamma_3^{\mathsf{Lie}}(\mathfrak{q}) = 0$ as $\mathfrak{q}$ is Lie-nilpotent of class 2. So $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) \subseteq \mathsf{Ker}(f)$. Therefore $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) \subseteq K(\mathfrak{q})$, since f is arbitrary.

For the reverse inclusion, we proceed by contradiction. Let $x \in K(\mathfrak{q})$ such that $x \notin \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$, and let $\mathfrak{h}$ be the complement of $\langle x \rangle$ in $\mathfrak{q}$. Then $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{q}$. So either $\mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{h}$ or $\mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$. The latter is not possible. Indeed, if $\mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$, then $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \gamma_2^{\mathsf{Lie}}(\mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q})) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{h}) + \gamma_3^{\mathsf{Lie}}(\mathfrak{q})$. But since $\mathfrak{q}$ is a Lie-nilpotent Leibniz algebra of class 2, it gives $\gamma_3^{\mathsf{Lie}}(\mathfrak{q}) = 0$, which implies that $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \gamma_2^{\mathsf{Lie}}(\mathfrak{h})$, and thus $\mathfrak{q} = \mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{h}) = \mathfrak{h}$, a contradiction. So we have $\mathfrak{h} + \gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{h}$, and thus $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) \subseteq \mathfrak{h}$, which implies that $\mathfrak{h}$ is a two-sided ideal of $\mathfrak{q}$. Now, choose $q_0 \in \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$, and consider the mapping $f : \mathfrak{q} \to \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$ defined by $h + \alpha x \mapsto \alpha q_0$. Clearly, f is a well-defined homomorphism of Leibniz algebras, and $\mathsf{Ker}(f) = \mathfrak{h}$. This is a contradiction, since $x \in K(\mathfrak{q})$ and $x \notin \mathfrak{h}$. Thus $K(\mathfrak{q}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$. $\square$

Theorem 3.10. *Let $\mathfrak{q}$ be a* Lie*-nilpotent Leibniz algebra of class 2. Then*

$$Z\left(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})\right) \cong T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}\right).$$

Proof. By the proof of Corollary 3.8, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ has a central subalgebra $\mathfrak{n}$ isomorphic to $T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}\right)$, where $\mathfrak{n} := \{d^* \mid d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})\}$ for some Lie-stem Leibniz algebra $\mathfrak{g}$ Lie-isoclinic to $\mathfrak{q}$. Denote this Lie-isoclinism by (η, ξ).

It remains to show that $Z\left(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})\right) \subseteq \mathfrak{n}$, that is, given $T \in Z\left(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})\right)$, we must find $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ such that $T = d^*$.

First, we claim that $T(\mathfrak{q}) \subseteq K(\mathfrak{q})$. Indeed, let $f : \mathfrak{q} \to [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$ be a homomorphism of Leibniz algebras. Then consider the mapping $t_f : \mathfrak{q} \to \mathfrak{q}$ defined by $t_f(z) = f(z)$. Clearly, $t_f \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$, since $t_f(\mathfrak{q}) \subseteq [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{q})$ as $\mathfrak{q}$ is a Lie-nilpotent Leibniz algebra of class 2. So $[T, t_f] = 0$, and thus $f(T(z)) = t_f(T(z)) = T(t_f(z)) = 0$ for all $z \in \mathfrak{q}$, since $t_f(z) \in [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$ and $T([\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}) = 0$ as $T \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$. Therefore $T(\mathfrak{q}) \subseteq \mathsf{Ker}(f)$. Hence $T(\mathfrak{q}) \subseteq K(\mathfrak{q})$, since f is arbitrary, which proves the claim.

It follows from Lemma 3.9 that $T(\mathfrak{q}) \subseteq [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}$. Now, for any $x \in \mathfrak{g}$, we have $x + Z_{\mathsf{Lie}}(\mathfrak{g}) = \eta^{-1}(y + Z_{\mathsf{Lie}}(\mathfrak{q}))$ for some $y \in \mathfrak{q}$, since η is bijective. Consider the map $d : \mathfrak{g} \to \mathfrak{g}$ defined by $x \mapsto \xi^{-1}(T(y))$. Clearly, d is well-defined, and it is easy to show that $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, since $T(\mathfrak{q}) \subseteq [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{q})$. Hence $T = d^*$. This completes the proof. □

Corollary 3.11. *Let $\mathfrak{q}$ be a finite dimensional* Lie*-nilpotent Leibniz algebra of class 2. Then $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ is abelian if and only if $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = Z_{\mathsf{Lie}}(\mathfrak{q})$.*

PROOF. Assume that $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = Z_{\mathsf{Lie}}(\mathfrak{q})$, then by Proposition 3.2, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ is abelian, since $\mathfrak{q}$ is a Lie-stem Leibniz algebra. Conversely, suppose that $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})$ is an abelian Lie algebra. Then, again by Proposition 3.2, $\mathfrak{q}$ is a Lie-stem Leibniz algebra. This implies by Lemma 3.5 that $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q}) \cong T\left(\frac{\mathfrak{q}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{q})}, Z_{\mathsf{Lie}}(\mathfrak{q})\right)$. Also, by Theorem 3.10, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q}) = Z\left(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})\right) \cong T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, \gamma_2^{\mathsf{Lie}}(\mathfrak{q})\right)$. It follows that $T\left(\frac{\mathfrak{q}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{q})}, Z_{\mathsf{Lie}}(\mathfrak{q})\right) \cong T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, \gamma_2^{\mathsf{Lie}}(\mathfrak{q})\right)$. Now, let K be the $\mathbb{K}$-vector subspace complement of $Z_{\mathsf{Lie}}(\mathfrak{q})$ in $\gamma_2^{\mathsf{Lie}}(\mathfrak{q})$. We claim that $K = 0$. Indeed, since as vector spaces $Z_{\mathsf{Lie}}(\mathfrak{q}) \oplus K = \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$, it holds

$$T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, \gamma_2^{\mathsf{Lie}}(\mathfrak{q})\right) = T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, Z_{\mathsf{Lie}}(\mathfrak{q})\right) \oplus T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, K\right).$$

As $\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})} \twoheadrightarrow \frac{\mathfrak{q}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{q})}$ by the Snake Lemma, it follows that $T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, \gamma_2^{\mathsf{Lie}}(\mathfrak{q})\right) \cong T\left(\frac{\mathfrak{q}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{q})}, Z_{\mathsf{Lie}}(\mathfrak{q})\right)$ is isomorphic to a subalgebra of $T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, Z_{\mathsf{Lie}}(\mathfrak{q})\right)$. Hence $T\left(\frac{\mathfrak{q}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{q})}, K\right) = 0$. This completes the proof. □

Example 3.12. The following is an example of a Leibniz algebra satisfying the requirements of Corollary 3.11.

Let $\mathfrak{q}$ be the three-dimensional Leibniz algebra with basis $\{a_1, a_2, a_3\}$ and bracket operation given by $[a_2, a_2] = [a_3, a_3] = a_1$ and zero elsewhere (see algebra 2 (c) in [9]). It is easy to check that $\gamma_2^{\mathsf{Lie}}(\mathfrak{q}) = Z_{\mathsf{Lie}}(\mathfrak{q}) =< \{a_1\} >$.

4. Lie-central derivations and Lie-centroids

Definition 4.1. The Lie-centroid of a Leibniz algebra $\mathfrak{g}$ is the set of all linear maps $d : \mathfrak{g} \to \mathfrak{g}$ satisfying the identities

$$d([x, y])_{\mathrm{lie}} = [d(x), y]_{\mathrm{lie}} = [x, d(y)]_{\mathrm{lie}}$$

for all $x, y \in \mathfrak{g}$. We denote this set by $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$.

Proposition 4.2. *For any Leibniz algebra* $\mathfrak{g}$, $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$ *is a subalgebra of* $\mathsf{End}(\mathfrak{g})$ *such that* $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}) \cap \Gamma^{\mathsf{Lie}}(\mathfrak{g})$.

Proof. Assume that $d \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}) \cap \Gamma^{\mathsf{Lie}}(\mathfrak{g})$. For all $x, y \in \mathfrak{g}$, we have that $d([x,y]_{\mathrm{lie}}) = [d(x), y]_{\mathrm{lie}} + [x, d(y)]_{\mathrm{lie}}$; on the other hand, $d([x,y]_{\mathrm{lie}}) = [x, d(y)]_{\mathrm{lie}}$, hence $[d(x), y]_{\mathrm{lie}} = 0$ for any $y \in \mathfrak{g}$, that is $d(x) \in Z_{\mathsf{Lie}}(\mathfrak{g})$.

Conversely, $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is a subalgebra of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, and for any $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, we have $d([x,y]_{\mathrm{lie}}) = [d(x), y]_{\mathrm{lie}} + [x, d(y)]_{\mathrm{lie}} = 0$, since $[d(x), y]_{\mathrm{lie}} = 0$, $[x, d(y)]_{\mathrm{lie}} = 0$, for any $x, y \in \mathfrak{g}$, hence $d \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$. □

Proposition 4.3. *Let* $\mathfrak{g}$ *be a Leibniz algebra. For any* $d \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ *and* $\phi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$, *the following statements hold:*

(a) $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}) \subseteq N_{\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})}(\Gamma^{\mathsf{Lie}}(\mathfrak{g}))$.

(b) $d \circ \phi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ *if and only if* $\phi \circ d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.

(c) $d \circ \phi \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ *if and only if* $[d, \phi] \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.

Proof. (a) Straightforward verification.

(b) Assume $d \circ \phi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$. Then

$$\begin{aligned}
[\phi, d]([x,y]_{\mathrm{lie}}) &= (\phi \circ d)([x,y]_{\mathrm{lie}}) - (d \circ \phi)([x,y]_{\mathrm{lie}}) \\
&= [(\phi \circ d)(x), y]_{\mathrm{lie}} + [x, (\phi \circ d)(y)]_{\mathrm{lie}} - [(d \circ \phi)(x), y]_{\mathrm{lie}} \\
&= [[\phi, d](x), y]_{\mathrm{lie}} + [x, (\phi \circ d)(y)]_{\mathrm{lie}} \\
&= [\phi, d]([x,y]_{\mathrm{lie}}) + [x, (\phi \circ d)(y)]_{\mathrm{lie}}.
\end{aligned}$$

Therefore $[x, (\phi \circ d)(y)]_{\mathrm{lie}} = 0$. Similarly, $[(d \circ \phi)(x), y]_{\mathrm{lie}} = 0$.

Conversely, assume $\phi \circ d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Then $[d, \phi]([x,y]_{\mathrm{lie}}) = (d \circ \phi)([x,y]_{\mathrm{lie}}) - (\phi \circ d)([x,y]_{\mathrm{lie}})$, hence $(d \circ \phi)([x,y]_{\mathrm{lie}}) = [d, \phi]([x,y]_{\mathrm{lie}})$, since $(\phi \circ d)([x,y]_{\mathrm{lie}}) = 0$. Now it is a routine task to check that $[d, \phi] \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$, which completes the proof.

(c) Assume $d \circ \phi \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$. A direct computation shows that $[\phi, d] \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$. On the other hand, it is easy to check that $[d, \phi] \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$, therefore $[\phi, d] = -[d, \phi] \in \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \cap \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$. Proposition 4.2 completes the proof.

Conversely, assume $[d, \phi] \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, then $(d \circ \phi)([x,y]_{\mathrm{lie}}) = [d, \phi]([x,y]_{\mathrm{lie}}) + (\phi \circ d)([x,y]_{\mathrm{lie}}) = (\phi \circ d)([x,y]_{\mathrm{lie}})$. Now it is easy to check that $\phi \circ d$ is a Lie-derivation of $\mathfrak{g}$. □

Definition 4.4. Let $\mathfrak{m}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{g}$. Then $\mathfrak{m}$ is said to be $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$-invariant if $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for all $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$.

Proposition 4.5. *Let $\mathfrak{g}$ be a Leibniz algebra and $\mathfrak{m}$ be a two-sided ideal of $\mathfrak{g}$. The following statements hold:*

(a) $C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m},0)$ *is invariant under* $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$.

(b) *Every* Lie*-perfect two-sided ideal* $\mathfrak{m}\,(\mathfrak{m}=\gamma_2^{\mathsf{Lie}}(\mathfrak{m}))$ *of* $\mathfrak{g}$ *is invariant under* $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$.

Proof. (a) Let $g \in C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m},0)$ and $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$, then $\varphi(g) \in C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m},0)$, since $[\varphi(g), m]_{\mathrm{lie}} = \varphi[g,m]_{\mathrm{lie}} = 0$, for all $m \in \mathfrak{m}$.

(b) Let $\mathfrak{m}$ be a Lie-perfect two-sided ideal of $\mathfrak{g}$ and let $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$. Then any $x \in \mathfrak{m}$ can be written as $x = \sum\limits_{i=1}^{n} \lambda_i [m_{i1}, m_{i2}]_{\mathrm{lie}}, m_{i1}, m_{i2} \in \mathfrak{m}$, hence $\varphi(x) = \sum\limits_{i=1}^{n} \lambda_i [\varphi(m_{i1}), m_{i2}]_{\mathrm{lie}} \in \mathfrak{m}$. □

Theorem 4.6. *Let $\mathfrak{m}$ be a nonzero $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$-invariant two-sided ideal of a Leibniz algebra $\mathfrak{g}$, $V(\mathfrak{m}) = \{\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \mid \varphi(\mathfrak{m}) = 0\}$ and $W = \mathsf{Hom}\left(\frac{\mathfrak{g}}{\mathfrak{m}}, C_{\mathfrak{g}}^{\mathsf{Lie}}(\mathfrak{m},0)\right)$. Define*

$$T(\mathfrak{m}) = \{f \in W \mid f[\overline{x},\overline{y}]_{\mathrm{lie}} = [f(\overline{x}),\overline{y}]_{\mathrm{lie}} = [\overline{x}, f(\overline{y})]_{\mathrm{lie}}\}$$

with $\overline{x} = x + \mathfrak{m}$ and $\overline{y} = y + \mathfrak{m}$. Then the following statements hold:

(a) *$T(\mathfrak{m})$ is a vector subspace of W isomorphic to $V(\mathfrak{m})$.*

(b) *If $\Gamma^{\mathsf{Lie}}(\mathfrak{m}) = \mathbb{K}.\mathsf{Id}_{\mathfrak{m}}$, then $\Gamma^{\mathsf{Lie}}(\mathfrak{g}) = \mathbb{K}.\mathsf{Id}_{\mathfrak{g}} \oplus V(\mathfrak{m})$ as vector spaces.*

Proof. (a) Define $\alpha : V(\mathfrak{m}) \longrightarrow T(\mathfrak{m})$ by $\alpha(\varphi)(x+\mathfrak{m}) = \varphi(x)$.

Obviously, α is an injective, well-defined linear map and it is onto. Indeed, for every $f \in T(\mathfrak{m})$, set $\varphi_f : \mathfrak{g} \longrightarrow \mathfrak{g}$, $\varphi_f(x) = f(x+\mathfrak{m})$, for all $x \in \mathfrak{g}$. It is easy to check that $\varphi_f \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ and $\varphi_f(\mathfrak{m}) = 0$, so $\varphi_f \in V(\mathfrak{m})$. Moreover, $\alpha(\varphi_f)(x+\mathfrak{m}) = \varphi_f(x) = f(x+\mathfrak{m})$.

(b) If $\Gamma^{\mathsf{Lie}}(\mathfrak{m}) = \mathbb{K}.\mathsf{Id}_{\mathfrak{m}}$, then for all $\psi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$, we get $\psi_{|\mathfrak{m}} = \lambda.\mathsf{Id}_{\mathfrak{m}}$, for some $\lambda \in \mathbb{K}$.

If $\psi \neq \lambda.\mathsf{Id}_{\mathfrak{g}}$, define $\varphi : \mathfrak{g} \to \mathfrak{g}$ by $\varphi(x) = \lambda x$, then $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ and $\psi - \varphi \in V(\mathfrak{m})$. Clearly, $\psi = \varphi + (\psi - \varphi) \in \mathbb{K}.\mathsf{Id}_{\mathfrak{g}} + V(\mathfrak{m})$. Furthermore, it is evident that $\mathbb{K}.\mathsf{Id}_{\mathfrak{g}} \cap V(\mathfrak{m}) = 0$, which completes the proof. □

Corollary 4.7. *If $\mathbb{K}$ is a field of characteristic zero, then the following equalities hold:*

$$\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) = V(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = T(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})).$$

Proof. If $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, then $d \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}) \cap \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ by Proposition 4.2, so $[d(x), y]_{\mathrm{lie}} = [x, d(y)]_{\mathrm{lie}} = 0$, hence $d \in V(\gamma_2^{\mathsf{Lie}}(\mathfrak{g}))$.

Conversely, if $d \in V(\gamma_2^{\mathsf{Lie}}(\mathfrak{g}))$, then $d \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ and $d([x,y]_{\mathrm{lie}}) = 0$, so $d([x,y]_{\mathrm{lie}}) = [d(x), y]_{\mathrm{lie}} = [x, d(y)]_{\mathrm{lie}} = 0$. Hence $d([x,y]_{\mathrm{lie}}) = [d(x), y]_{\mathrm{lie}} + [x, d(y)]_{\mathrm{lie}} = 0$, which implies that $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.

The second equality is provided by Theorem 4.6, since $\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ is $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$-invariant. $\square$

Theorem 4.8. *Let $\mathfrak{g}$ be a Leibniz algebra such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$, $\mathfrak{g}_2$ are two-sided ideals of $\mathfrak{g}$. Then the following isomorphism of $\mathbb{K}$-vector spaces holds:*

$$\Gamma^{\mathsf{Lie}}(\mathfrak{g}) \cong \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1) \oplus \Gamma^{\mathsf{Lie}}(\mathfrak{g}_2) \oplus C_1 \oplus C_2,$$

where $C_i = \{\varphi \in \mathsf{Hom}(\mathfrak{g}_i, \mathfrak{g}_j) \mid \varphi(\mathfrak{g}_i) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g}_j)$ and $\varphi(\gamma_2^{\mathsf{Lie}}(\mathfrak{g}_i)) = 0$ for $1 \leq i \neq j \leq 2\}$.

Proof. Let $\pi_i : \mathfrak{g} \longrightarrow \mathfrak{g}_i$ be the canonical projection for $i = 1, 2$. Then $\pi_1, \pi_2 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ and $\pi_1 + \pi_2 = \mathsf{Id}_{\mathfrak{g}}$.

So we have for $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ that $\varphi = \pi_1 \circ \varphi \circ \pi_1 + \pi_1 \circ \varphi \circ \pi_2 + \pi_2 \circ \varphi \circ \pi_1 + \pi_2 \circ \varphi \circ \pi_2$.

Note that $\pi_i \circ \varphi \circ \pi_j \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ for $i, j = 1, 2$. So, by the above equality it follows that

$$\Gamma^{\mathsf{Lie}}(\mathfrak{g}) = \pi_1 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_1 \oplus \pi_1 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_2 \oplus \pi_2 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_1 \oplus \pi_2 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_2$$

as vector spaces. Indeed, it is enough to show that $\pi_i \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_k \cap \pi_l \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_j = 0$ for $i, j, k, l = 1, 2$, such that $(i, j) \neq (k, l)$. For instance, $\pi_2 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_1 \cap \pi_1 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_2 = 0$, since for any $\beta \in \pi_2 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_1 \cap \pi_1 \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_2$, there are some $f_1, f_2 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ such that $\beta = \pi_2 \circ f_1 \circ \pi_1 = \pi_1 \circ f_2 \circ \pi_2$, and then $\beta(x) = \pi_1 \circ f_2 \circ \pi_2(x) = \pi_1 \circ f_2 \circ \pi_2(\pi_2(x)) = \pi_2 \circ f_1 \circ \pi_1(\pi_2(x)) = \pi_2 \circ f_1(0) = 0$, for all $x \in \mathfrak{g}$. Hence $\beta = 0$. Other cases can be checked in a similar way.

Now put $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{ij} = \pi_i \Gamma^{\mathsf{Lie}}(\mathfrak{g}) \pi_j$, $i, j = 1, 2$. We claim that the following isomorphisms of vector spaces hold:

$$\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11} \cong \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1),\ \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{22} \cong \Gamma^{\mathsf{Lie}}(\mathfrak{g}_2),\ \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12} \cong C_2,\ \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{21} \cong C_1.$$

For $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11}$, we have $\varphi(\mathfrak{g}_2) = 0$, so $\varphi_{|\mathfrak{g}_1} \in \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1)$. Now, considering $\Gamma^{\mathsf{Lie}}(\mathfrak{g}_1)$ as a subalgebra of $\Gamma^{\mathsf{Lie}}(\mathfrak{g})$ such that for any $\varphi_0 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1)$, φ_0 vanishes on $\mathfrak{g}_2$, that is, $\varphi_0(x_1) = \varphi_0(x_2)$, $\varphi_0(x_2) = 0$, for all $x_1 \in \mathfrak{g}_1$ and $x_2 \in \mathfrak{g}_2$. Then $\varphi_0 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ and $\varphi_0 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11}$. Therefore, $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11} \cong \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1)$ by means of the isomorphism $\sigma : \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11} \longrightarrow \Gamma^{\mathsf{Lie}}(\mathfrak{g}_1)$, $\sigma(\varphi) = \varphi_{|\mathfrak{g}_1}$, for all $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{11}$.

The isomorphism $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{22} \cong \Gamma^{\mathsf{Lie}}(\mathfrak{g}_2)$ can be proved in an analogous way.

We now prove that $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12} \cong C_2$. Indeed, for any $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12}$, there exists a $\varphi_0 \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})$ such that $\varphi = \pi_1 \circ \varphi_0 \circ \pi_2$. For $x_k = (x_k^1, x_k^2) \in \mathfrak{g}$, where $x_k^i \in \mathfrak{g}_i$, $i = 1, 2$, $k = 1, 2$, we have

$$\begin{aligned}\varphi\left([x_1, x_2]_{\mathrm{lie}}\right) &= \pi_1 \circ \varphi_0 \circ \pi_2\left([x_1, x_2]_{\mathrm{lie}}\right) = \pi_1 \circ \varphi_0 \circ \pi_2\left([(x_1^1, x_1^2), (x_2^1, x_2^2)]_{\mathrm{lie}}\right)\\ &= \pi_1\varphi_0\left([x_1^2, x_2^2]_{\mathrm{lie}}\right) = \pi_1\left([\varphi_0\left(x_1^2\right), x_2^2]_{\mathrm{lie}}\right) = 0,\end{aligned}$$

hence $\varphi(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = 0$. On the other hand, $[\varphi(x_1), x_2]_{\mathrm{lie}} = \varphi([x_1, x_2]_{\mathrm{lie}}) = 0$, so, $\varphi(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$ and $\varphi(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = 0$.

It follows that $\varphi_{|\mathfrak{g}_2}(\mathfrak{g}_2) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g}_1)$ and $\varphi_{|\mathfrak{g}_2}(\gamma_2^{\mathsf{Lie}}(\mathfrak{g}_2)) = 0$, hence $\varphi_{|\mathfrak{g}_2} \in C_2$.

Conversely, for $\varphi \in C_2$, expanding φ on $\mathfrak{g}$ by $\varphi(\mathfrak{g}_1) = 0$, we have $\pi_1 \circ \varphi \circ \pi_2 = \varphi$, and so $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12}$. Hence $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12} \cong C_2$, by means of the isomorphism $\tau : \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12} \longrightarrow C_2$, $\tau(\varphi) = \varphi_{|\mathfrak{g}_2}$ for all $\varphi \in \Gamma^{\mathsf{Lie}}(\mathfrak{g})_{12}$.

Similarly, it can be proved that $\Gamma^{\mathsf{Lie}}(\mathfrak{g})_{21} \cong C_1$, which completes the proof. □

5. $\mathsf{ID}^{\mathsf{Lie}}$-derivations

Definition 5.1. A Lie-derivation $d : \mathfrak{g} \to \mathfrak{g}$ is said to be an ID-Lie-derivation if $d(\mathfrak{g}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$. The set of all ID-Lie-derivations of $\mathfrak{g}$ is denoted by $\mathsf{ID}^{\mathsf{Lie}}(\mathfrak{g})$.

An ID-Lie-derivation $d : \mathfrak{g} \to \mathfrak{g}$ is said to be an ID_*-Lie-derivation if d vanishes on the Lie-central elements of $\mathfrak{g}$. The set of all ID_*-Lie-derivations of $\mathfrak{g}$ is denoted by $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$.

It is obvious that $\mathsf{ID}^{\mathsf{Lie}}(\mathfrak{g})$ and $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ are subalgebras of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ and

$$\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}) \subseteq \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) \subseteq \mathsf{ID}^{\mathsf{Lie}}(\mathfrak{g}), \tag{4}$$

where $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ is the subspace of $\mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g})$ given by $\{d \in \mathsf{Der}^{\mathsf{Lie}}(\mathfrak{g}) \mid d(x) \in [x, \mathfrak{g}]_{\mathrm{lie}}, \forall x \in \mathfrak{g}\}$. These kinds of derivations are called almost inner Lie-derivations of $\mathfrak{g}$.

Example 5.2. Let $\mathfrak{g}$ be the three-dimensional Leibniz algebra with basis $\{a_1, a_2, a_3\}$ and bracket operation given by $[a_2, a_2] = [a_3, a_3] = a_1$ and zero elsewhere (algebra 2 (c) in [9]). The right multiplication Lie-derivations $R_x, x \in \mathfrak{g}$, are examples of almost inner Lie-derivations.

Definition 5.3. An almost inner Lie-derivation d is said to be a central almost inner Lie-derivation if there exists an $x \in Z^l(\mathfrak{g})$ such that $(d - R_x)(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$.

We denote the $\mathbb{K}$-vector space of all central almost inner Lie-derivations by $\mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$.

Theorem 5.4. *Let $\mathfrak{g}$ and $\mathfrak{q}$ be two* Lie*-isoclinic Leibniz algebras. Then* $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) \cong \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$.

Proof. Let (η, ξ) be the Lie-isoclinism between $\mathfrak{g}$ and $\mathfrak{q}$, and let $\alpha \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$. Consider the map $\zeta_\alpha : \mathfrak{q} \to \mathfrak{q}$ defined by $\zeta_\alpha(y) := \xi(\alpha(x))$, where $y + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x + Z_{\mathsf{Lie}}(\mathfrak{g}))$. Clearly, ζ_α is a well-defined linear map, since α and ξ are linear maps, and if $y \in Z_{\mathsf{Lie}}(\mathfrak{q})$, then $x \in Z_{\mathsf{Lie}}(\mathfrak{g})$, and thus $\zeta_\alpha(y) = \xi(\alpha(x)) = \zeta(0) = 0$. To show that ζ_α is a Lie-derivation, let $y_1, y_2 \in \mathfrak{q}$ and $x_1, x_2 \in \mathfrak{g}$ such that $y_i + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x_i + Z_{\mathsf{Lie}}(\mathfrak{g}))$, $i = 1, 2$. Then

$$\begin{aligned}
\zeta_\alpha([y_1, y_2]_{\mathrm{lie}}) &= \xi(\alpha([x_1, x_2]_{\mathrm{lie}})) \\
&= \xi([\alpha(x_1), x_2]_{\mathrm{lie}}) + \xi([x_1, \alpha(x_2)]_{\mathrm{lie}}) \quad \text{by [3, Prop. 3.8]} \\
&= [\xi(\alpha(x_1)), y_2]_{\mathrm{lie}} + [y_1, \xi(\alpha(x_2))]_{\mathrm{lie}} \\
&= [\zeta_\alpha(y_1), y_2]_{\mathrm{lie}} + [y_1, \zeta_\alpha(y_2)]_{\mathrm{lie}}.
\end{aligned}$$

Moreover, since $\alpha(\mathfrak{g}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ and ξ is an isomorphism, it follows that $\zeta_\alpha(\mathfrak{q}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{q})$. Therefore $\zeta_\alpha \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$. Now consider the map $\zeta : \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) \to \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$ defined by $\zeta(\alpha) = \zeta_\alpha$. We claim that ξ is a Lie-homomorphism. Indeed, for $\alpha_1, \alpha_2 \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$, we have for all $y \in \mathfrak{q}$ and $x \in \mathfrak{g}$ such that $y + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x + Z_{\mathsf{Lie}}(\mathfrak{g}))$,

$$\begin{aligned}
\zeta([\alpha_1, \alpha_2])(y) &= \zeta_{[\alpha_1, \alpha_2]}(y) = \xi([\alpha_1, \alpha_2](x)) = \xi(\alpha_1(\alpha_2(x)) - \alpha_2(\alpha_1(x))) \\
&= \xi(\alpha_1(\alpha_2(x))) - \xi(\alpha_2(\alpha_1(x))) \\
&= \zeta_{\alpha_1}(\xi(\alpha_2(x)) - \zeta_{\alpha_2}(\xi(\alpha_1(x)) \quad \text{by [3, Prop. 3.8]} \\
&= \zeta_{\alpha_1}(\zeta_{\alpha_2}(y)) - \zeta_{\alpha_2}(\zeta_{\alpha_1}(y)) = [\zeta_{\alpha_1}, \zeta_{\alpha_2}](y) = [\zeta(\alpha_1), \zeta(\alpha_2)](y).
\end{aligned}$$

Hence $\zeta([\alpha_1, \alpha_2]) = [\zeta(\alpha_1), \zeta(\alpha_2)]$. Conversely, let $\beta \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$. By using the inverse Lie-isoclinism (η^{-1}, ξ^{-1}), we similarly construct a homomorphism $\zeta' : \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q}) \to \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ defined by $\zeta'(\beta) = \zeta'_\beta$, where $\zeta'_\beta(x) = \xi^{-1}(\beta(y))$ with $y + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x + Z_{\mathsf{Lie}}(\mathfrak{g}))$. It is clear that $(\zeta' \circ \zeta)(\alpha)(x) = \zeta'(\zeta(\alpha))(x) = \zeta'_{\zeta(\alpha)}(x) = \xi^{-1}(\zeta(\alpha)(y)) = \xi^{-1}(\zeta_\alpha(y)) = \xi^{-1}(\xi(\alpha(x))) = \alpha(x)$. So $\zeta' \circ \zeta = \mathsf{Id}_{\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})}$. Similarly, one shows that $\zeta \circ \zeta' = \mathsf{Id}_{\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})}$. Therefore $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) \cong \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})$. □

Corollary 5.5. *Let $\mathfrak{g}$ and $\mathfrak{q}$ be two* Lie*-isoclinic Leibniz algebras. Then* $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}) \cong \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})$.

Proof. Let (η, ξ) be the Lie-isoclinism between $\mathfrak{g}$ and $\mathfrak{q}$, and let $\alpha \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$. Consider again the map $\zeta_\alpha : \mathfrak{q} \to \mathfrak{q}$ defined by $\zeta_\alpha(y) := \xi(\alpha(x))$, where $y + Z_{\mathsf{Lie}}(\mathfrak{q}) = \eta(x + Z_{\mathsf{Lie}}(\mathfrak{g}))$, given in the proof of Theorem 5.4. Since $\alpha(x) \in [x, \mathfrak{g}]_{\mathrm{lie}}$

and ξ is an isomorphism, it is clear that $\zeta_\alpha(y) \in [y, \mathfrak{q}]_{\mathrm{lie}}$ for all $y \in \mathfrak{q}$. So $\zeta_\alpha \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})$. So the restriction $\zeta_{|\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})} : \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}) \to \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})$ of the map ζ in the proof of Theorem 5.4 to $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ is a homomorphism. Similarly, by using the inverse Lie-isoclinism (η^{-1}, ξ^{-1}), one obtains a homomorphism by taking the restriction $\zeta'_{|\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})} : \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q}) \to \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ of the map ζ' in the proof of Theorem 5.4 to $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})$. It is clear that $\zeta \circ \zeta'_{|\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})} = \mathsf{Id}_{\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})}$ and $\zeta' \circ \zeta_{|\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})} = \mathsf{Id}_{\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})}$. Therefore $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}) \cong \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{q})$. □

For any $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, the map $\psi_d : \frac{\mathfrak{g}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{g})} \to Z_{\mathsf{Lie}}(\mathfrak{g})$ given by $\psi_d(g + \gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = d(g)$ is a linear map. It is easy to show that the linear map $\psi : \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}) \to T\left(\frac{\mathfrak{g}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{g})}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)$, $\psi(d) = \psi_d$, is bijective. Therefore, for any finite dimensional Leibniz algebra $\mathfrak{g}$, $\dim\left(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})\right) = \dim\left(T\left(\frac{\mathfrak{g}}{\gamma_2^{\mathsf{Lie}}(\mathfrak{g})}, Z_{\mathsf{Lie}}(\mathfrak{g})\right)\right)$.

Corollary 5.6. *Let $\mathfrak{g}$ be a finite dimensional Leibniz algebra such that $[\mathfrak{g}, \mathfrak{g}] = \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z^r(\mathfrak{g})$. Then $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ if and only if $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g})$.*

Proof. Assume that $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g})$. It is clear that for all $d \in \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, one has that $d(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$ if and only if $d(\mathfrak{g}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ and $d(Z_{\mathsf{Lie}}(\mathfrak{g})) = d(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = 0$. Therefore $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$.

Conversely, assume that $\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. Then, since $[\mathfrak{g}, \mathfrak{g}] = \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z^r(\mathfrak{g})$, it follows that the map $R_x : \mathfrak{g} \to \mathfrak{g}$, $R_x(y) = [y, x]$, is a Lie-derivation. Moreover, it is easy to check that $R_x \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$, hence $R_x(y) \in Z_{\mathsf{Lie}}(\mathfrak{g})$, for all $y \in \mathfrak{g}$. Therefore $\mathcal{Z}_2^{\mathsf{Lie}}(\mathfrak{g}) = \mathfrak{g}$, and thus $\mathfrak{g}$ is Lie-nilpotent of class 2 by Theorem 2.4. Now, by [5, Corollary 4.1], there is a Lie-stem Leibniz algebra $\mathfrak{q}$ Lie-isoclinic to $\mathfrak{g}$. Denote this Lie-isoclinism by (η, ξ). Since $\mathfrak{g}$ is a Lie-nilpotent Leibniz algebra of class 2, so is $\mathfrak{q}$, due to Lemma 3.7. This implies that $[\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}} \overset{\xi}{\cong} [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}} = Z_{\mathsf{Lie}}(\mathfrak{q})$, and $\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})} \overset{\eta}{\cong} \frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})} \cong \frac{\mathfrak{q}}{[\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}}$. It follows from Theorem 5.4, the first implication and Lemma 3.5 that

$$\begin{aligned}
\dim(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})) &= \dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})) = \dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{q})) = \dim(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{q})) \\
&= \dim\left(T\left(\frac{\mathfrak{q}}{[\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}}, Z_{\mathsf{Lie}}(\mathfrak{q})\right)\right) = \dim\left(T\left(\frac{\mathfrak{q}}{Z_{\mathsf{Lie}}(\mathfrak{q})}, [\mathfrak{q}, \mathfrak{q}]_{\mathsf{Lie}}\right)\right) \\
&= \dim\left(T\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}, [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}\right)\right) = \dim\left(Z(\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g}))\right).
\end{aligned}$$

The latter equality is due to Theorem 3.10, since $\mathfrak{g}$ is Lie-nilpotent of class 2. Therefore $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$ is abelian. We now conclude by Corollary 3.11 that $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g})$. □

Remark 5.7. Let us observe that the requirements $[\mathfrak{g},\mathfrak{g}]=\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ and $Z_{\mathsf{Lie}}(\mathfrak{g})\subseteq Z^r(\mathfrak{g})$ in Corollary 5.6 are not needed in the absolute case, but in our relative setting they are absolutely necessary as the following counterexample shows. Let $\mathfrak{g}$ be the four-dimensional complex Leibniz algebra with basis $\{a_1,a_2,a_3,a_4\}$ and bracket operation given by $[a_1,a_2]=-[a_2,a_1]=a_4;[a_3,a_3]=a_4$ and zero elsewhere (class $\mathfrak{R}_{21}$ in [1, Theorem 3.2]). It is easy to check that $[\mathfrak{g},\mathfrak{g}]=\langle\{a_4\}\rangle=\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$, $Z_{\mathsf{Lie}}(\mathfrak{g})=\langle\{a_1,a_2,a_4\}\rangle$ and $Z^r(\mathfrak{g})=\langle\{a_4\}\rangle$.

Consider the Lie-derivation R_{a_1}, which belongs to $\mathsf{Der}_z^{\mathsf{Lie}}(\mathfrak{g})$. However, $R_{a_1}\notin\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$, since R_{a_1} does not vanish on $Z_{\mathsf{Lie}}(\mathfrak{g})$.

Example 5.8. The three-dimensional complex Leibniz algebra with basis $\{a_1,a_2,a_3\}$ and bracket operation given by $[a_2,a_2]=\gamma a_1,\gamma\in\mathbb{C};[a_3,a_2]=[a_3,a_3]=a_1$ and zero elsewhere (class 2 (a) in [9]) satisfies the requirements of Corollary 5.6, since $[\mathfrak{g},\mathfrak{g}]=\gamma_2^{\mathsf{Lie}}(\mathfrak{g})=Z_{\mathsf{Lie}}(\mathfrak{g})=Z^r(\mathfrak{g})=\langle\{a_1\}\rangle$.

Theorem 5.9. *Let $\mathfrak{g}$ be a Leibniz algebra such that $\gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ is finite dimensional and $\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}$ is generated by p elements. Then*

$$\dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}))\leq p\cdot\dim(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})).$$

Proof. Consider the map $\alpha:\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})\to T\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})},\gamma_2^{\mathsf{Lie}}(\mathfrak{g})\right)$ defined by $d\mapsto d^*$ such that $d^*(x+Z_{\mathsf{Lie}}(\mathfrak{g}))=d(x)$. Then α is a well-defined injective linear map. It follows that $\dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}))\leq\dim\left(T\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})},\gamma_2^{\mathsf{Lie}}(\mathfrak{g})\right)\right)=p\cdot\dim(\gamma_2^{\mathsf{Lie}}(\mathfrak{g}))$ □

Example 5.10. Now we present two examples illustrating the inequality in Theorem 5.9.

(a) Let $\mathfrak{g}$ be the three-dimensional Leibniz algebra with basis $\{a_1,a_2,a_3\}$ and bracket operation given by $[a_2,a_3]=-[a_3,a_2]=a_2,[a_3,a_3]=a_1$ and zero elsewhere (class 2 (f) in [9]).

It is an easy task to check that $\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}=\langle\{\overline{a}_3\}\rangle$, hence the number of generators is $p=1$. Moreover, $\gamma_2^{\mathsf{Lie}}(\mathfrak{g})=\langle\{a_1\}\rangle$. Also, it can be checked that an element $d\in\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ is represented by a matrix of the form

$$\begin{pmatrix}0&0&a_{13}\\0&0&0\\0&0&0\end{pmatrix}.$$

Hence $\dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g}))=1\leq 1\cdot 1.$

(b) Let $\mathfrak{g}$ be the four-dimensional Leibniz algebra with basis $\{a_1, a_2, a_3, a_4\}$ and bracket operation given by $[a_1, a_4] = a_1, [a_2, a_4] = a_2$ and zero elsewhere (class $\mathcal{R}_2$ in [7, Theorem 2.7]).

It is an easy task to check that $\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})} = \langle\{\overline{a}_1, \overline{a}_2, \overline{a}_4\}\rangle$, hence the number of generators is $p = 3$. Moreover, $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) = \langle\{a_1, a_2\}\rangle$. Also, it can be checked that an element $d \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ is represented by a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\dim(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})) = 4 \leq 3 \cdot 2$.

Corollary 5.11. *Let $\mathfrak{g}$ be a Leibniz algebra such that $Z^r(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g})$, $[\mathfrak{g}, \mathfrak{g}] = \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$ is finite dimensional and $\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}$ is generated by p elements. Then*

$$\dim\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}\right) \leq p \cdot \dim(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})).$$

Proof. Under these hypotheses, we have from the proof of Corollary 5.6 that $R_x \in \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. Now, consider the $\mathbb{K}$-linear map $\beta : \frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})} \to \mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})$ defined by $x + Z_{\mathsf{Lie}}(\mathfrak{g}) \mapsto R_x$, which is an injective well-defined linear map, since $\mathsf{Ker}(\beta) = \frac{Z^r(\mathfrak{g})}{Z_{\mathsf{Lie}}(\mathfrak{g})} = 0$. Hence $\dim\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}\right) \leq \dim\left(\mathsf{ID}_*^{\mathsf{Lie}}(\mathfrak{g})\right)$. Now Theorem 5.9 completes the proof. □

Example 5.12. The three-dimensional non-Lie Leibniz algebra with basis $\{a_1, a_2, a_3\}$ and bracket operation $[a_3, a_3] = a_1$ and zero elsewhere (class 2 (b) in [9]) satisfies the requirements of Corollary 5.11.

Definition 5.13. A Leibniz algebra $\mathfrak{g}$ of dimension n is said to be Lie-filiform (or 1-Lie-filiform) if $\dim(\gamma_i^{\mathsf{Lie}}(\mathfrak{g})) = n - i$, $2 \leq i \leq n$.

Lie-filiform Leibniz algebras are Lie-nilpotent Leibniz algebras of class $n - 1$.

Corollary 5.14. *Let $\mathfrak{g}$ be an n-dimensional Leibniz algebra such that $Z^r(\mathfrak{g}) = Z_{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z^l(\mathfrak{g})$ and it attains the upper bound of Corollary 5.11. If $\mathfrak{g}$ is* Lie-*filiform, then $n = 3$.*

Proof. If $\mathfrak{g}$ is Lie-filiform, then $\dim(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = n - 2,\ n \geq 2$. Then we have $p = \dim\left(\frac{\mathfrak{g}}{Z_{\mathsf{Lie}}(\mathfrak{g})}\right) = p \cdot \dim(\gamma_2^{\mathsf{Lie}}(\mathfrak{g})) = p(n-2)$, which implies that $n = 3$. □

Remark 5.15. Example 5.12 provides a Lie-filiform Leibniz algebra which illustrates Corollary 5.14.

Proposition 5.16. *Let $\mathfrak{g}$ be a Leibniz algebra. Then the following statements hold:*

(a) *Let $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$. Then $d(\mathfrak{g}) \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$, $d(Z_{\mathsf{Lie}}(\mathfrak{g})) = 0$ and $d(\mathfrak{n}) \subseteq \mathfrak{n}$ for every two-sided ideal $\mathfrak{n}$ of $\mathfrak{g}$.*

(b) *For $d \in \mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$, there exists an $x \in Z^l(\mathfrak{g})$ such that $d_{|\gamma_2^{\mathsf{Lie}}(\mathfrak{g})} = R_{x|\gamma_2^{\mathsf{Lie}}(\mathfrak{g})}$.*

(c) *If $\mathfrak{g}$ is 2-step Lie-nilpotent, then $\mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g}) = \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$.*

(d) *If $Z_{\mathsf{Lie}}(\mathfrak{g}) = 0$, then $\mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g}) \subseteq \mathsf{R}(\mathfrak{g})$ and $\mathsf{R}(Z^l(\mathfrak{g})) \subseteq \mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$.*

(e) *If $\mathfrak{g}$ is Lie-nilpotent, then $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ is Lie-nilpotent and all $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ are nilpotent.*

(f) *$\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g} \oplus \mathfrak{g}') = \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}) \oplus \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}')$, for any Leibniz algebras $\mathfrak{g}$ and $\mathfrak{g}'$.*

PROOF. (a) For any $x \in \mathfrak{g}$, we have $d(x) \in [x, \mathfrak{g}]_{\mathsf{Lie}} \subseteq [\mathfrak{g}, \mathfrak{g}]_{\mathsf{Lie}}$; if $x \in Z_{\mathsf{Lie}}(\mathfrak{g})$, then $d(x) = [x, y]_{\mathrm{lie}} = 0$, for all $y \in \mathfrak{g}$; $d(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{n}$.

(b) Let $d \in \mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$, then there exists $x \in Z^l(\mathfrak{g})$ such that $(d - R_x)(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$. Since $d - R_x$ is a Lie-derivation, we have

$$(d - R_x)([y, z]_{\mathrm{lie}}) = [(d - R_x)(y), z]_{\mathrm{lie}} + [y, (d - R_x)(z)]_{\mathrm{lie}} = 0,$$

and thus $d([y, z]_{\mathrm{lie}}) = R_x([y, z]_{\mathrm{lie}})$, for all $y, z \in \mathfrak{g}$. Hence $d_{|\gamma_2^{\mathsf{Lie}}(\mathfrak{g})} = R_{x|\gamma_2^{\mathsf{Lie}}(\mathfrak{g})}$.

(c) Notice that if $\mathfrak{g}$ is 2-step Lie-nilpotent, then $\gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$. So for all $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$, any $x \in Z^l(\mathfrak{g})$ and $y \in \mathfrak{g}$, we have $d(y) \in [y, \mathfrak{g}]_{\mathsf{Lie}} \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$ and $R_x(y) = [y, x] = [y, x]_{\mathrm{lie}} \in \gamma_2^{\mathsf{Lie}}(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$. Therefore $(d - R_x)(\mathfrak{g}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$, and thus $d \in \mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$.

(d) Assume that $Z_{\mathsf{Lie}}(\mathfrak{g}) = 0$. Then for all $d \in \mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g})$, there exists an $x \in Z^l(\mathfrak{g})$ such that $(d - R_x)(\mathfrak{g}) = 0$, i.e., $d = R_x \in \mathsf{R}(\mathfrak{g})$. So $\mathsf{Der}_{cz}^{\mathsf{Lie}}(\mathfrak{g}) \subseteq \mathsf{R}(\mathfrak{g})$. The second inclusion can be easily checked.

(e) If $\mathfrak{g}$ is Lie-nilpotent of class c, then $\gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g}) = 0$. So for any $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$, $d(x) \in [x, \mathfrak{g}]_{\mathsf{Lie}} \subseteq \gamma_2^{\mathsf{Lie}}(\mathfrak{g})$. One inductively proves that $d^c(x) \in \gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g})$, $d^c(x) = d(d^{c-1}(x)) \in [d^{c-1}(x), \mathfrak{g}]_{\mathsf{Lie}} \subseteq \gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g}) = 0$. So d is nilpotent.

Also, a routine inductive argument shows that $\gamma_{c+1}^{\mathsf{Lie}}(\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}))(\mathfrak{g}) \subseteq \gamma_{c+1}^{\mathsf{Lie}}(\mathfrak{g}) = 0$. So $\gamma_{c+1}^{\mathsf{Lie}}(\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})) = 0$, and thus $\mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ is Lie-nilpotent.

(f) For any $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g} \oplus \mathfrak{g}')$, it is clear that $d_{|\mathfrak{g}} \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ and $d_{|\mathfrak{g}'} \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}')$. Conversely, for $d \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g})$ and $d' \in \mathsf{Der}_c^{\mathsf{Lie}}(\mathfrak{g}')$, one easily shows that the mapping $d'' : \mathfrak{g} \oplus \mathfrak{g}' \to \mathfrak{g} \oplus \mathfrak{g}'$ defined by $d''(x, x') := (d(x), d'(x'))$ is a Lie-derivation such that for $(x, x') \in \mathfrak{g} \oplus \mathfrak{g}'$, we have $d''(x, x') = (d(x), d'(x')) \in ([x, \mathfrak{g}]_{\mathsf{Lie}}, [x', \mathfrak{g}']_{\mathsf{Lie}}) = [(x, x'), \mathfrak{g} \oplus \mathfrak{g}']_{\mathsf{Lie}}$ by definition of the bracket of $\mathfrak{g} \oplus \mathfrak{g}'$. $\square$

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